Lecture 9

In the last lecture, we learned the concept of symmetric points and the way to find imaging circle under linear fractional transformation. In this lecture, we study the orientation of a given circle.

1. Using z_2, z_3 and z_4, we can determine a unique circle passing across these three points. We denote this circle by C. If z is on the circle, then we have $\mathrm{Im}(z, z_2, z_3, z_4) = 0$.

2. C automatically separate the complex plane into two parts. One part contains all z where $\mathrm{Im}(z, z_2, z_3, z_4) < 0$. We call this part the algebraic left-hand side of the circle C with respect to the triple (z_2, z_3, z_4). Another part contains all z where $\mathrm{Im}(z, z_2, z_3, z_4) > 0$. We call this part the algebraic right-hand side of C with respect to the triple (z_2, z_3, z_4).

3. The definition in 2 for the left and right-hand side of C corresponding to the triple (z_2, z_3, z_4) is an algebraic way to describe the side for a given circle. Here is a geometric way to understand it. If the triple (z_2, z_3, z_4) is given, then we can decide a unique direction on the circle C so that by following this direction we can go from z_2 to z_3 and then to z_4 in order. One can easily see that the direction that we can have is just counterclockwise or clockwise direction. But once (z_2, z_3, z_4) is given in order, then the counterclockwise or clockwise direction is uniquely fixed so that along this direction we go from z_2 to z_3 and then to z_4 in order. Clearly if you are moving along counterclockwise direction, then the interior part of circle C is on your left. This corresponds to the left-side defined in 2 where $\mathrm{Im}(z, z_2, z_3, z_4) < 0$. Meanwhile the exterior part of C is on your right. This is the right side defined in 2 where $\mathrm{Im}(z, z_2, z_3, z_4) > 0$. But if you are moving clockwisely, then the situation is different. Now you can see that exterior part of C is on your left which corresponds to the region where $\mathrm{Im}(z, z_2, z_3, z_4) < 0$, while the interior part is on your right which corresponds to the region where $\mathrm{Im}(z, z_2, z_3, z_4) > 0$.

4. One may want a rigorous argument for the facts stated in 3. Here it is. Fixing a triple (z_2, z_3, z_4), we know that there is unique direction so that we can go from z_2 to z_3 and then to z_4 in order. Suppose that the direction is counterclockwise (see graph). The situation when direction is clockwise can be similarly treated. Firstly we assume z is an arbitrary point in the interior part of the circle C. Connecting z_3 and z_4, we get a line l. This line l separates the interior of circle C into two parts. One part contains the point z_2 and another part does not contain z_2. Therefore the position of z can be classified into three cases.

Case 1. z and z_2 are on the same side of l In this case we see that we can rotate $z - z_3$ counterclockwisely by an angle α so that the new vector has the same direction as $z - z_4$. Clearly $\alpha \in (0, \pi)$. Moreover we can also rotate $z_2 - z_3$ counterclockwisely by an angle β so that the new vector has the same direction as $z_2 - z_4$. Clearly $\beta \in (0, \pi)$. Fundamental geometry tells us that $\alpha > \beta$. Therefore in Case 1, we have $0 < \beta < \alpha < \pi$. We can calculate that

$$\begin{aligned} \mathrm{Im}(z, z_2, z_3, z_4) &= \mathrm{Im}\left(\frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}\right) \\ &= \mathrm{Im}\left(\frac{|z - z_3||z_2 - z_4|}{|z - z_4||z_2 - z_3|} e^{i\left[(\arg(z - z_3) - \arg(z - z_4)) - (\arg(z_2 - z_3) - \arg(z_2 - z_4))\right]}\right). \end{aligned} \tag{0.1}$$

Moreover by the above arguments (also see the graph for case 1), we have $\arg(z - z_3) + \alpha = \arg(z - z_4)$ and $\arg(z_2 - z_3) + \beta = \arg(z_2 - z_4)$. Applying these two equalities to (0.1), we get

$$\mathrm{Im}(z, z_2, z_3, z_4) = \frac{|z - z_3||z_2 - z_4|}{|z - z_4||z_2 - z_3|} \sin(\beta - \alpha).$$

Since now $\alpha - \beta \in (0, \pi)$, it holds $\sin(\beta - \alpha) < 0$. Therefore $\mathrm{Im}(z, z_2, z_3, z_4) < 0$ for case 1;

Case 2. z is on l. In this case one can show that α in case 1 equals to π. Therefore $\sin(\beta - \pi) = -\sin\beta < 0$. We still have $\mathrm{Im}(z, z_2, z_3, z_4) < 0$.

Case 3. z is on the side without z_2. In this case $\alpha \in (\pi, 2\pi)$. Moreover from the graph for case 3, we have $\gamma_1 > \gamma_2$, $\gamma_1 + \alpha = 2\pi$ and $\gamma_2 + \beta = \pi$. Therefore we know that $2\pi - \alpha > \pi - \beta$ which show that $\alpha - \beta < \pi$. Of course we have $\alpha - \beta > 0$. This tells us that $\sin(\beta - \alpha) < 0$ which still shows that $\mathrm{Im}(z, z_2, z_3, z_4) < 0$.

From the above arguments we know that for all z in the interior part of the circle C, $\mathrm{Im}(z, z_2, z_3, z_4) < 0$. Now we consider exterior points. Suppose that w is a point on the exterior part of the circle C. Then its symmetric point w^* with respect to the circle C must be in the interior part of C. By the previous arguments, we know that $\mathrm{Im}(w^*, z_2, z_3, z_4) < 0$. Since $(w^*, z_2, z_3, z_4) = \overline{(w, z_2, z_3, z_4)}$, therefore it holds $\mathrm{Im}(w, z_2, z_3, z_4) = -\mathrm{Im}(w^*, z_2, z_3, z_4) > 0$. In summary, we know that if the direction is counterclockwise, then the geometric left-side (the left if you are moving counterclockwisely, i.e. interior part of C) coincides with the algebraic left-side (the side where $\mathrm{Im}(z, z_2, z_3, z_4) < 0$). Moreover the geometric right-side (the right if you are moving counterclockwisely, i.e. exterior part of C) coincides with the algebraic right-side (the side where $\mathrm{Im}(z, z_2, z_3, z_4) > 0$). If we include the clockwise case in our consideration, we then have

Proposition 0.1. *Given a triple (z_2, z_3, z_4) on C, we can find a direction on C so that by following this direction, we go from z_2 to z_3 and then to z_4 in order. The geometric right-hand side of C coincide with the algebraic right-hand side of C. The geometric left-hand side of C coincides with the algebraic left-hand side of C.*

With the above proposition and the fact that cross ratio is invariant under linear transformations, we can show that

Proposition 0.2. *Linear transformations map left-hand (right-hand) side to left-hand (right-hand) side.*

Remark 0.3. *Proposition 0.2 should be understood as follows. given (z_2, z_3, z_4) a triple on a circle C, we can decide a direction on C. Given an arbitrary linear transformation T, the triple (z_2, z_3, z_4) is sent to (Tz_2, Tz_3, Tz_4) which decide a direction for the imaging circle of C. Therefore Proposition 0.2 tells us that the left side of C with respect to the direction given by (z_2, z_3, z_4) coincides with the left side of the imaging circle of C with respect to the direction given by (Tz_2, Tz_3, Tz_4).*

Proof of Proposition 0.2. If C is determined by z_2, z_3 and z_4 and the direction of the circle C is given by the triple (z_2, z_3, z_4), then the imaging circle is determined by Tz_2, Tz_3 and Tz_3. Here T is a linear transformation. Moreover if we go from z_2 to z_3 and then to z_4 in order, then in the imaging circle we can induce a direction which let us go from Tz_2 to Tz_3 and then to Tz_4 in order. (z_2, z_3, z_4) decide a direction for C. (Tz_2, Tz_3, Tz_4) decide a direction for the image of C. If z is on the left of C, then $\mathrm{Im}(z, z_2, z_3, z_4) < 0$. Therefore $\mathrm{Im}(Tz, Tz_2, Tz_3, Tz_4) = \mathrm{Im}(z, z_2, z_3, z_4) < 0$. This tells us that Tz is on the left of the imaging circle of C whose direction is given by the triple (Tz_2, Tz_3, Tz_4). The proof is finished since the right-side case can be similarly treated. □

Now we begin to study some other elementary functions. One of the most important elementary functions is the exponential function.

Definition 0.4. *Given $z = x + iy$, we denote by e^z the exponential function with $e^z = e^x(\cos y + i \sin y)$.*

With the above definition, we can remark that

Remark 0.5. *$|e^z| = e^x$, which depends only on the variable x.*

Remark 0.6. *e^z is a periodic function with period $2k\pi i$.*

Remark 0.7. *e^z is not a function defined on the Riemann sphere. One can show that as $x \longrightarrow -\infty$, $e^x \longrightarrow 0$; As $x \longrightarrow \infty$, $e^x \longrightarrow \infty$. The two limits are different. So e^z is not well defined at ∞.*

Remark 0.8. *e^z is derivable. Furthermore one can calculate that $(e^z)' = \partial_x u + i\partial_x v = e^x \cos y + ie^x \sin y = e^z$.*

With the definition of e^z, we can introduce a sort of so-called elementary transcendental functions. They are

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}.$$

We can also define the so-called trigo functions. they are

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Clearly sinh and cosh functions have period $2k\pi i$, while sin and cos functions have period $2k\pi$. These functions are all well-defined on $\mathbb{C}$ but not on the Riemann sphere.

We now begin to study the inverse function of e^z. Letting $z = \rho e^{i\theta}$, we assume $w = w_1 + iw_2$ so that $z = e^w$. Clearly we have

$$\rho e^{i\theta} = e^{w_1} e^{iw_2}. \tag{0.2}$$

Taking absolute value on both sides above, we get $\rho = e^{w_1}$. Therefore it holds $w_1 = \log \rho = \log |z|$. Plugging this w_1 into (0.2), we know that

$$e^{i\theta} = e^{iw_2}.$$

Since cos and sin are periodic functions with period equaling to 2π, it holds $w_2 = \theta + 2k\pi$ where k is an integer. Therefore we know that

$$w = \log |z| + i(\arg(z) + 2k\pi).$$

But $\arg(z)$ is not uniquely decided. So we define $\mathrm{Arg}(z)$ which is called principal argument and takes values in $[-\pi, \pi)$. With the principal argument, we know that

$$w = \log |z| + i(\mathrm{Arg}(z) + 2k\pi), \qquad k \in \mathbb{Z} \tag{0.3}$$

gives us all solutions of (0.2). For an inverse function of e^z, there is only one valued assigned to each z. In other words, we can only choose one value from (0.3) to define an inverse function of e^z. Therefore we need a rule to decide a unique k in (0.3). An easy way to do so is to assign for each z a restriction function $\alpha(z)$. This $\alpha(z)$ is real valued and it is used to restrict the imaginary part of w in (0.3) within the interval $[\alpha(z), \alpha(z) + 2\pi)$. With this $\alpha(z)$, we know that we can fix a unique $k \in \mathbb{Z}$ so that $\mathrm{Arg}(z) + 2k\pi \in [\alpha(z), \alpha(z) + 2\pi)$. Therefore this value can be used to define an inverse function of e^z.

Example 1: $\alpha(z) \equiv \pi/4$. In this case, $w_2(z)$ takes its value in $[\pi/4, \pi/4 + 2\pi)$, for all $z \in \mathbb{C}$. If $z = i$, then we know that $\mathrm{Arg}(i) = \pi/2$. If we want $\pi/2 + 2k\pi \in [\pi/4, \pi/4 + 2\pi)$, then $k = 0$. This tells us that if the restriction function $\alpha(z) \equiv \pi/4$, then $\log i = \pi/2$.

Example 2: Find $\log i$ with $\alpha(z) = 3\pi/4$.
Solution: $\alpha(z) \equiv 3\pi/4$ implies that $w_2(z) \in [3\pi/4, 3\pi/4 + 2\pi)$. If we want $\pi/2 + 2k\pi \in [3\pi/4, 3\pi/4 + 2\pi)$, then $k = 1$. Therefore in this case, $\log i = i(\pi/2 + 2\pi)$.